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GEOMETRIC INTEGRATION THEORY

By

Hassler Whitney

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Preface

In various branches of mathematics and its applications, in particular, in differential geometry and in physics, one has often to integrate a quantity over an r-dimensional manifold M in n-space E^n, for instance, over a surface in ordinary 3-space. Considering M (or a portion of M) as the image of part of r-space E^r, integration over M is reduced to integration in E^r, where standard theory applies. However, it is important to know in what manner the integral over M depends on the position of M in E^n, assuming the quantity to be integrated is defined throughout a region R containing M. Thus we must consider the integral over M as a function of the position of M in E^n. The main purpose of this book is to study this function, in a broad geometric and analytic setting.

Starting with Chapter V, we use a postulational approach. Assuming the simplest properties of what r-dimensional integration in n-space should be like, we are led to a theory which turns out to be precisely the integration of differential forms, which may be of a very general character. Hence the role of differential forms in integration theory is more firmly fixed, and at the same time the scope of the theory is considerably increased.

The subject requires an understanding of the geometric properties of the "direction" of an r-dimensional element in n-space, and of course the fundamentals of calculus in several dimensions. The classical treatment, using coordinate systems, results in sometimes lengthy formulas, which do not make the underlying geometric ideas clear, and whose parts depend on the coordinate system employed. Hence in the first part of the book we give a full exposition of this material, in an elementary manner. The geometric approach is gradually coming into use at present; it is hoped that the early chapters may help in making the methods accessible to the general reader.

An overall picture of what the book is about may be obtained from the introductory chapter; we show how the simplest hypotheses lead to the basic tools employed, and we illustrate these tools particularly in the 3-dimensional case. For a more complete outline of results, one may read the introductory pages to the different chapters. Preliminary material that is somewhat outside the scope of the study but is needed in various parts of the book is collected in the appendices.

The body of the book falls into three Parts. The first Part, Classical theory, leads up to the theory of the Riemann integral; we include also a study of smooth (i.e. differentiable) manifolds. The early chapters should be accessible to the beginning graduate student. The second Part, General theory, gives a postulational approach. More maturity on the part of the reader is assumed here. In the last Part, we continue the general study, using Lebesgue theory.

Except where we consider smooth manifolds, we remain always in Euclidean n-dimensional space E^n. Since we use normed spaces as a tool, the metric of E^n is employed; however, the fundamental geometric ideas and theorems are independent of the metric. The study of smooth manifolds in Chapter IV lies outside the main stream of thought of the book; it is included because of its wide interest and application. We prove de Rham's Theorem by elementary means (much like de Rham's original proof). In this manner, the theorem lies close to a theorem deriving the cohomology properties of a complex from abstract integration theory; see § 12 of Chapter VII.

For other expositions bearing some relation to our Part I, see the books mentioned below of Bourbaki (for Chapter I), Lichnerowicz (for Chapters I, II and III), and de Rham (for Chapter IV).

We now give a brief description of our approach to the general problem of r-dimensional integration in n-space. An "integral" is something defined over oriented r-dimensional cells, and over linear combinations of cells; it becomes a function of polyhedral r-chains. This function is linear, and hence we call it a "cochain." We define two norms in the linear space of polyhedral chains, the "flat" norm $|A|^\flat$ and the "sharp" norm $|A|^\sharp$. The cochains which are bounded functions in one of these norms are "flat" or "sharp" correspondingly. A straightforward proof shows that a sharp cochain corresponds to a differential form, in that the value of the cochain on any polyhedral chain equals the integral of the form over the chain. Thus the theory of a certain class of differential forms is derived from the simplest assumptions about the integral. The similar theorem in the flat case is due to Wolfe.

In the case of n-dimensional integration in n-space, the "flat" theory is equivalent to the Lebesgue theory of bounded measurable functions. To obtain all locally summable functions, one should define a space of cochains more general than is possible through a norm. We know of no conditions expressible simply in the r-dimensional case which lead to differential forms in all cases, and to all measurable locally summable functions in the n-dimensional case. In the introductory pages of Chapter V we give conditions leading to arbitrary continuous forms. See also the introductory pages of Chapter VIII.

The domains of integration are always oriented, for $r > 0$; we have $\int_{-A} \omega = -\int_A \omega$, or in the terminology of cochains, $X\cdot(-A) = -X\cdot A$. In theories where orientation properties play no role, we prefer to consider the integration as 0-dimensional; see the last section of the Introduction. Since the spaces of polyhedral chains have been given norms, we may "integrate" over any element of the completions of the spaces, that is, over any "flat chain" or "sharp chain". One type of flat chain is given by an oriented (curved) piece of an r-dimensional manifold in E^n. We show this in very general circumstances in Chapter X. Another type of flat chain is determined by a continuous summable function in E^n whose values are r-vectors; see § 7 of Chapter VI (also § 25 of the Introduction). In this case, an apparently n-dimensional integral may be interpreted as an r-dimensional integral.

The differential forms ω coming from flat cochains are "flat" forms. They are measurable functions satisfying two boundedness conditions. Using properties of flat cochains, the standard properties of forms are derived; the exterior differential $d\omega$ exists (though it may not be definable through differentiation), as does the image $f^*\omega$ of ω if f is a Lipschitz mapping. Hence also cohomology with real coefficients in polyhedra (and in Lipschitz spaces, see below) may be studied through flat forms (as in de Rham's Theorem). In particular, the cup product of flat forms is anti-commutative, and obeys the other standard relations for products of cochains.

The "mass" of flat and sharp chains is definable. In the last chapter, we study the structure of sharp chains A of finite mass. One may find the "part A_Q of A" in any Borel set Q. Generalizing the notion of the r-vector of an oriented r-cell, one may define the r-vector $\{B\}$ of any r-chain B. Now with A given, $\Phi(Q) = \{A_Q\}$ is an additive function of Borel sets in E^n, whose values are r-vectors. It is shown that this set function characterizes the chain A. The theory of these chains now becomes the theory of these set functions.

In the theory of distributions and currents in a manifold (due to L. Schwartz and G. de Rham, see the book of de Rham quoted below), one starts with a simple space of forms (cochains), and obtains the currents (which include smooth singular chains) as linear functions on the forms with some continuity conditions. We do just the reverse: starting with chains, cochains are obtained as linear functions. Because of this, the spaces of cochains and chains obtained are quite different from the above spaces of forms and currents; the explicit study of our chains and cochains has little relation to the standard theory of currents.

On the other hand, starting with smooth singular chains in a Riemannian manifold and introducing sequences of semi-norms, one may obtain directly by a limiting process various spaces of currents and distributions. This procedure, offering certain advantages over the usual one, has been given by James Eells, Jr., in Geometric aspects of currents and distributions, *Proc. Nat. Ac. of Sci. 41* (1955), 493–496.

This book had its beginnings in a study of integration in "Lipschitz spaces"; see Algebraic topology and integration theory, *Proc. Nat. Ac. of Sci. 33* (1947), 1–6. The theory in Euclidean space was at that time restricted principally to the case of sharp cochains (there called "tensor cochains"). The discovery by J. H. Wolfe, in *Tensor Fields Associated with Lipschitz Cochains*, Harvard Thesis, 1948, that flat cochains (then called Lipschitz cochains) correspond to differential forms, caused a fundamental change in the point of view. The study in Euclidean space now became primary; having found that the theory of cochains could be built on norms of chains, these norms became the basic tool. For reasons of space, Lipschitz spaces were finally dropped out of the book; the author expects to give the theory for this case in a separate memoir. An account of the more recent state of the work (including results on Lipschitz spaces) may be found in *r-dimensional integration in n-space*, Proc. International Congress of Mathematicians, Providence, 1952. A study of the Riemann integral in a geometric manner similar to the early chapters of the book was made by Paul Olum in a Senior Thesis at Harvard in 1940.

A choice of notations was sometimes difficult, owing to the overlapping of the fields of integration theory and algebraic topology. Since the operations of exterior differentiation of forms and of taking the coboundary of cochains coalesce in the present work, a single symbol should be used; we finally chose the d of analysis rather than the δ of topology. We use ∇ in place of d for the ordinary differential to avoid confusion with the above d. The symbols $\vee$ and $\wedge$ chosen for the products in Grassmann algebra correspond exactly to the usual symbols $\smile$ and $\frown$ in topology; see Chapter IX.

We shall have occasion to refer to the following texts. This will be done by mention of the author's name.

Banach, S. *Théorie des operations linéaires*, Subwencji funduszu kultury narodowej, Warszawa, 1932.

Bourbaki, N. *Éléments de mathématique*, Livre II, Algèbre, Chapitre III, Hermann, Paris, 1948.

Halmos, P. *Measure Theory*, Van Nostrand, 1950.

Lichnerowicz, A. *Algèbre et analyse linéaires*, Masson, Paris, 1947.

de Rham, G. *Variétés différentiables*, Hermann, Paris, 1955.

Saks, S. *Theory of the Integral*, Subwencji funduszu kultury narodowej, Warszawa, 1937.

A reference such as (V, 10.4) applies to equation (4) in § 10 of Chapter V; (App. I, 7) means § 7 of Appendix I.

The author wishes to acknowledge the permission of J. H. Wolfe to incorporate the results of his thesis into the book. He is indebted to Norman Z. Wolfsohn for much help in the manuscript, and to James Eells for collaboration in revising Chapter XI, for other help in the manuscript, and for aid in proofreading.

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Part I: Classical theory

Introduction

The purpose of this preliminary chapter is first of all to provide motivation for the methods and tools appearing in the book, and secondly to illustrate some of the general considerations through a study of special cases, particularly in three dimensions. The Introduction and the body of the book are independent of each other; however, the meaning of the full theory will become clearer if the Introduction is read in conjunction with the rest of the book.

In Part A (which is rather abstract in character), we ask what a theory of r-dimensional integration in n-space should look like. An "integral" $X \cdot \sigma$ is defined over an oriented r-cell σ for instance, and changes sign if the orientation is reversed. We may now define $X \cdot (a\sigma) = a(X \cdot \sigma)$ for real numbers a, and $X \cdot (A + B) = X \cdot A + X \cdot B$; assuming that a subdivision of cells does not affect the integral, we now have a linear function defined over "polyhedral r-chains." To give this linear function analytical properties, we introduce some continuity hypotheses. Next we study the local nature of integration. Near any point, in any r-dimensional direction, the integral over a cell is approximately proportional to the r-dimensional volume of the cell (with the strongest continuity hypothesis); this fact leads to the construction of a point function $D_X(p)$ which acts on "r-vectors" $\{\sigma\}$ of oriented r-cells σ. These r-vectors must have certain simple properties, which in turn lead directly to the construction of Grassmann algebra. Finally, D_X becomes a differential r-form, whose integral $\int_\sigma D_X$ over any σ equals the original $X \cdot \sigma$.

In Part B, we start with the elements of Grassmann algebra as derived above, and work out from a geometric point of view some of the fundamentals of calculus. We consider vector analysis in three dimensions, differentials, Jacobians, transformation of "multiple integrals," manifolds, and the Theorems of Stokes and de Rham.

The purpose of Part C is to introduce the reader to some of the general methods in the later parts of the book. In the last two sections, we touch on the way some particular modes of integration may be considered as r-dimensional for different r.

A. The General Problem of Integration

1. The integral as a function of the domain. For an integration theory there must certainly be various possible "domains of integration."

Whatever kind of process integration (with real values) is, a definite "integrand" X will give a real number when applied to a permissible domain A. Thus, for a fixed X, we have a real valued function of domains A; we denote the value on A by $X \cdot A$. We consider integration in Euclidean space.

If we are to call the integration r-dimensional, we must certainly include among the permissible domains the simplest r-dimensional figures. An r-cell σ, consisting of a closed bounded part of an r-plane, bounded by a finite number of pieces of $(r-1)$-planes, is such a figure. We make our first hypothesis:

HYPOTHESIS (H_1). The integral over σ depends on the orientation of σ; a reversal of orientation reverses the sign of the integral.

We discuss the meaning of and reason for this hypothesis. A line segment σ^1 has two end points p and q; the two orientations of σ^1 are the two directions along σ^1, from p to q and from q to p; we may denote the two oriented cells by $pq = -qp$ and by $qp = -pq$ respectively. They may be defined by a choice of the vector $q - p$ (from p to q) or $p - q$ (from q to p). We may orient a triangle $p_0p_1p_2$ by choosing an ordered pair of independent vectors in it; for instance, the pair $(p_1 - p_0, p_2 - p_0)$. Interchanging these or reversing the direction of either would reverse the orientation. Similarly, an r-cell σ is oriented by the choice of an ordered set of r independent vectors in it. A 0-cell, i.e. a single point, has no orientation properties.

The triangle $\sigma = p_0p_1p_2$, oriented as above, has a boundary $\partial\sigma$, consisting of the oriented segments p_0p_1, p_1p_2, and p_2p_0. The boundary $\partial(pq)$ of the oriented segment pq consists of the point q counted positively, and the point p counted negatively. The boundary $\partial\sigma$ of an r-cell σ contains its $(r-1)$-faces, properly oriented (App. II, 5).

For any oriented cell σ, let $-\sigma$ denote the oppositely oriented cell; then (H_1) may be written in the form

$$X \cdot (-\sigma) = -X \cdot \sigma. \tag{1}$$

We consider some examples. Let ϕ be a real valued function defined in 3-space E^3, and let C be an oriented curve, from the point p to the point q. If we integrate the rate of change of ϕ along C, we obtain $\phi(q) - \phi(p)$. If the orientation of C were reversed, we would obtain $\phi(p) - \phi(q)$. Next, consider any 1-dimensional integral $\int_A \omega$. Let $\sigma = p_0p_1p_2$ be an oriented triangle, cut into the two oriented triangles $\sigma' = p_0p'p_2$, $\sigma'' = p'p_1p_2$, by the segment $p'p_2$, with p' in p_0p_1. Then

$$\int_{\partial\sigma'} \omega + \int_{\partial\sigma''} \omega = \int_{\partial\sigma} \omega; \tag{2}$$

for

$$\int_{p_0p'} \omega + \int_{p'p_1} \omega = \int_{p_0p_1} \omega, \qquad \int_{p'p_2} \omega + \int_{p_2p'} \omega = 0,$$

the latter since $p'p_2$ and p_2p' are oppositely oriented. This type of relation is fundamental for geometric properties of the integral; it would be impossible if orientation properties were disregarded.

Of course integration over non-oriented domains is possible and sometimes of importance; in this case, much more general types of domains are permissible, but the geometric properties are largely lost. We prefer to think of such integration as 0-dimensional; see § 26 below.

The most typical requirement of integration theory is additivity, or in the present exposition, *invariance under subdivisions* (used in the example above):

HYPOTHESIS (H_2). If the oriented r-cell σ is cut into similarly oriented r-cells $\sigma_1, \cdots, \sigma_m$, then

$$X \cdot \sigma = X \cdot \sigma_1 + \cdots + X \cdot \sigma_m. \tag{3}$$

2. Polyhedral chains. We wish to write the boundary $\partial\sigma$ of the triangle $\sigma = p_0p_1p_2$ as a domain of integration. With the point p' in p_0p_1 as in § 1, we would like to write $\partial\sigma$ in various ways, such as

$$\partial\sigma = p_0p_1 + p_1p_2 + p_2p_0 = p_0p' + p'p_2 - p_0p_2 + p'p_1 + p_1p_2 - p'p_2$$

etc. This suggests the definition of a *polyhedral r-chain* A as being a linear combination of oriented r-cells, with real numbers as coefficients, together with the properties

$$1\sigma = \sigma, \qquad 0\sigma = 0, \qquad a(-\sigma) = (-a)\sigma = -(a\sigma), \tag{1}$$

and invariance under subdivision: if the oriented cell σ is cut into $\sigma_1, \cdots, \sigma_m$, then σ and $\sigma_1 + \cdots + \sigma_m$ are the same polyhedral chain. The definitions of aA (for real numbers a) and $A + B$ are obvious; the set of polyhedral r-chains now forms a linear space.

Because of (H_1) and (H_2), it is possible to define $X \cdot A$ for any polyhedral r-chain $A = \sum a_i\sigma_i$, by the relation

$$X \cdot \sum a_i\sigma_i = \sum a_i(X \cdot \sigma_i); \tag{2}$$

X is now a *linear function of polyhedral r-chains*. For this reason, we call X an *r-cochain*. That X is a cochain is equivalent to assuming that $X \cdot \sigma$ is defined, with the properties (H_1) and (H_2).

The boundary of $A = \sum a_i\sigma_i$ is defined to be $\partial A = \sum a_i\, \partial\sigma_i$; this is easily seen to be a well defined polyhedral $(r-1)$-chain. A polyhedral 0-chain is an expression $A^0 = \sum a_ip_i$, the p_i being points; we set $\partial A^0 = 0$.

3. Two continuity hypotheses. For a satisfactory integration theory, the permissible domains must include oriented curved r-cells for instance. One should be able to obtain these as limits of polyhedral r-chains, and the integral should be definable as the limit of the integrals over the approximating polyhedral chains. This requires some continuity hypotheses on the integral. We give two hypotheses in this section with which a satisfactory general theory may be obtained; if we include also the hypothesis of the next section, the integral has simpler analytical properties.

Let σ be an oriented triangle, of area $|\sigma|$. If we cut the plane containing σ into small rectangles and let $\tau_1, \cdots, \tau_m$ be those contained in σ, then these τ_i fill up most of σ, and it is natural to require that $\sum X \cdot \tau_i$ be near $X \cdot \sigma$. This will follow from:

HYPOTHESIS (H_1'). Given the r-cochain X, there is a number N_1 such that

(1) $$|X \cdot \sigma| \leqq N_1 |\sigma|, \qquad \text{all oriented } r\text{-cells } \sigma,$$

where $|\sigma|$ is the r-dimensional volume of σ.

This is of course a stronger hypothesis than needed for the above requirement. We assume it largely for the sake of the analytical methods described in § 6 below.

Take the case $r = 0$. We may consider the "0-dimensional volume" of a point p to be 1. Any 0-cochain X corresponds to a real function ϕ: $\phi(p) = X \cdot p$ for all points p. Now (H_1') says that $|\phi(p)| \leqq N_1$, all p; that is, ϕ is bounded. These functions are too general; we must restrict them further. We look for a hypothesis suggested by the case $r = 1$.

Take the above triangle σ again; let τ be the union of the τ_i. We may consider the boundary $\partial\tau$ of τ as an approximation to $\partial\sigma$, even though it is made up of segments not parallel to the sides of σ in general. Taking $r = 1$, we may require that $X \cdot \partial\tau$ be near $X \cdot \partial\sigma$, i.e. that $X \cdot \partial(\sigma - \tau)$ be small, as a result of the area of $\sigma - \tau$ being small.

HYPOTHESIS (H_2'). Given the r-cochain X, there is a number N_2 such that

(2) $$|X \cdot \partial\sigma^{r+1}| \leqq N_2 |\sigma^{r+1}|, \qquad \text{all oriented } (r+1)\text{-cells } \sigma^{r+1}.$$

Note that (H_2') is trivially satisfied if $r = n$.

For $r = 0$, (H_2') says that for any oriented segment pq,

(3) $$|X \cdot q - X \cdot p| = |X \cdot \partial(pq)| \leqq N_2 |q - p|,$$

$|q - p|$ being the length of pq; that is, the function $\phi(p) = X \cdot p$ satisfies a Lipschitz condition.

Any cochain satisfying (H_1') and (H_2') we call a *flat* cochain.

4. A further continuity hypothesis. If the cell σ is moved into a nearby position σ', we may assume that $X \cdot \sigma'$ is near $X \cdot \sigma$. We shall consider rigid motions without turning; that is, translations by means of vectors v. Let $T_v\sigma$ denote the new cell. Our hypothesis is that $X \cdot T_v\sigma$ differs from $X \cdot \sigma$ by at most some fixed multiple of the r-volume $|\sigma|$ times the distance $|v|$ of translation.

HYPOTHESIS (H_3'). Given the r-cochain X, there is a number N_3 such that for any oriented r-cell σ and vector v,

(1) $$|X \cdot T_v\sigma - X \cdot \sigma| \leqq N_3 |\sigma| |v|.$$

In the case $r = 0$, this hypothesis is equivalent to (H_2'). For $r = n$, it is non-trivial, whereas (H_2') is trivial.

Any cochain satisfying all three hypotheses we call *sharp*. We shall see that sharp cochains correspond to differential forms. (This holds also in the flat case; see Chapter IX.)

5. Some examples. We help elucidate some of the hypotheses through the study of a steady flow of fluid in oriented 3-space E^3. Take any oriented 2-cell σ. Let (v_1, v_2) define its orientation, and choose a vector v_3 so that (v_1, v_2, v_3) (or equivalently, (v_3, v_1, v_2)) defines the given orientation of E^3; then the *positive* direction through σ is the direction in the sense of v_3. Let $X \cdot \sigma$ be the quantity (positive or negative) of fluid flowing through σ in the positive direction in unit time; this is the *flux* across σ. Clearly (H_1) and (H_2) hold; hence X is a 2-cochain. Of course $X \cdot S$ for any oriented surface S is definable.

If the density of fluid and the velocity of flow are bounded, then clearly (H_1') holds. Now take any 3-cell τ. If fluid is being neither created nor destroyed, then (we assume the density constant in time) the total rate of flow out of τ, which equals $X \cdot \partial\tau$, must be 0. In general, $X \cdot \partial\tau$ equals the total rate of creation of fluid in τ. Hence (H_2') is equivalent to assuming that the total rate of creation per unit volume is bounded.

With the same flow of fluid, consider the *circulation* along an oriented curve C. At a point p of C, if $u(p)$ is the unit tangent vector at p in the positive direction along C, $v(p)$ is the velocity vector of the fluid at p, and $\rho(p)$ is the density at p, then the circulation is

(1) $$Y \cdot C = \int_C \rho v \cdot u$$

(compare (18.3) below). Again (H_1) and (H_2) hold.

Hypothesis (H_1') will follow from the boundedness of v and ρ. Given an oriented 2-cell σ, $Y \cdot \partial\sigma$ is the circulation around the boundary $\partial\sigma$. Taking arbitrarily small cells σ near a point p, we see that (H_2') will follow if curl $(v) = \nabla \times v$ exists and is finite; compare (21.4).

Suppose the flow is through a pipe. Then the above cochains X and Y are not defined throughout E^3, but only in the region of flow; the hypotheses need be assumed only in this region.

6. The case $r = n$. For an n-cochain X which satisfies (H_1') in oriented E^n ((H_2') is satisfied trivially), it is standard Lebesgue theory that there is a bounded measurable function Φ such that (using the Lebesgue integral)

(1) $$X \cdot \sigma = \int_\sigma \Phi, \qquad \text{all } n\text{-cells } \sigma^n \text{ oriented like } E^n.$$

We consider briefly the simpler case when (H_3') is also satisfied. Given the point p, let $\sigma_1, \sigma_2, \cdots$ be a sequence of n-cells oriented like E^n, in smaller and smaller neighborhoods of p. Set

$$\Phi(p) = \lim_{i\to\infty} \frac{X\cdot\sigma_i}{|\sigma_i|}. \tag{2}$$

We indicate the proof of existence and uniqueness of the limit. Let $\tau_1, \tau_2, \cdots$ be a similar sequence of cubes. Suppose i_0 and k_0 are such that for $i \geqq i_0$ and $k \geqq k_0$, σ_i and τ_k are in a neighborhood of p of diameter $<\epsilon$. We may take k so large that translations of τ_k nearly fill up σ_i:

$$\sigma_i = \sigma' + R, \qquad \sigma' = T_{v_1}\tau_k + \cdots + T_{v_s}\tau_k, \qquad |R| \text{ small.}$$

By (H_3'),

$$\left|\frac{X\cdot\sigma'}{|\sigma'|} - \frac{X\cdot\tau_k}{|\tau_k|}\right| \leqq \frac{1}{s}\sum_{j=1}^{s}\left|\frac{X\cdot T_{v_j}\tau_k - X\cdot\tau_k}{|\tau_k|}\right| \leqq N_3\epsilon,$$

and the statement follows, using (H_1').

Using (H_3') again, it is clear that Φ satisfies

$$|\Phi(p+v) - \Phi(p)| \leqq N_3|v|, \tag{3}$$

and that (1) holds (using the Riemann integral). Moreover,

$$|X\cdot\sigma - |\sigma|\Phi(p_0)| = \left|\int_\sigma [\Phi(p) - \Phi(p_0)]\,dp\right| \leqq N_3\zeta|\sigma| \tag{4}$$

if all points of σ are within ζ of p_0.

7. The r-vector of an oriented r-cell. Let X be a sharp r-cochain in E^n. Then for each oriented r-plane P in E^n, we may consider $X\cdot\sigma$ for r-cells in P, and hence find a function Φ_P in P as in § 6. For any point p, the values of $\Phi_P(p)$ for the various oriented r-planes P through p are of interest.

We shall use a closely related function. Given σ and p, let P be the r-plane through p parallel to σ and oriented like σ, and set

$$D_X(p)\cdot\{\sigma\} = |\sigma|\,\Phi_P(p). \tag{1}$$

We must give meaning to $D_X(p)$, to $\{\sigma\}$, and to their combination. With the σ_i as in § 6, (1) and (6.2) give

$$D_X(p)\cdot\{\sigma\} = \lim_{i\to\infty}\frac{|\sigma|}{|\sigma_i|}X\cdot\sigma_i. \tag{2}$$

As a function of σ, the right hand side is known as soon as we know the set of r-planes parallel with σ, the orientation of σ, and $|\sigma|$; we call this triple the *r-vector* $\{\sigma\}$ of σ. We may now define $D_X(p)$ to be that real valued function defined on all r-vectors of oriented r-cells which is given by (1). (Later $D_X(p)$ will be taken to be defined on more special spaces $\mathbf{T}_r$.)

We remark that $D_X(p)\cdot\{\sigma\}$ is unchanged if we alter the metric in E^n; for using (2), we see that $|\sigma|/|\sigma_i|$ is unchanged, as is $X\cdot\sigma_i$.

Given $\{\sigma\}$ and the real number $a \neq 0$, let $a\{\sigma\}$ denote $\{\sigma'\}$ for any oriented σ' parallel with σ, oriented like or opposite to σ according as $a > 0$ or $a < 0$, and such that $|\sigma'| = |a||\sigma|$. If σ_1 and σ_2 are parallel, say $\{\sigma_2\} = a\{\sigma_1\}$; set

$$\{\sigma_1\} + \{\sigma_2\} = (1+a)\{\sigma_1\},$$

for $a \neq -1$. If we include a "zero r-vector" 0, then with these definitions the r-vectors associated with a fixed set of parallel r-planes form a linear space isomorphic with the real numbers. In any such space of r-vectors, we see easily from (2) that $D_X(p)$ is linear:

$$D_X(p)\cdot(a\{\sigma\} + b\{\sigma'\}) = aD_X(p)\cdot\{\sigma\} + bD_X(p)\cdot\{\sigma'\}. \tag{3}$$

8. On r-vectors and boundaries of $(r+1)$-cells. Let us immerse the set of all r-vectors of oriented r-cells in a linear space $\mathbf{S}_r$ (of infinite dimension for $0 < r < n$) as follows. Take a fixed point p_0. An element α of $\mathbf{S}_r$ is a finite set of distinct r-planes $P_1, \cdots, P_m$ through p_0, together with an r-vector α_i associated with each P_i; we may include extra planes P_j, if we associate the zero r-vector with them. We form $a\alpha$ by replacing each α_i by $a\alpha_i$. Given α and β, we may take enough planes $P_1, \cdots, P_m$ so that α and β are defined by α_i and β_i in P_i respectively $(i = 1, \cdots, m)$; let $\alpha + \beta$ be defined by $\alpha_i + \beta_i$ in P_i. Clearly $\mathbf{S}_r$ is independent of the choice of p_0. The linear spaces described in the last section are linear subspaces of $\mathbf{S}_r$.

We may extend $D_X(p)$ to be a linear function in $\mathbf{S}_r$ by defining

$$D_X(p)\cdot\alpha = \sum_{i=1}^{m} D_X(p)\cdot\alpha_i \quad \text{if} \quad \alpha \text{ is defined by } \alpha_i \text{ in } P_i\ (i = 1, \cdots, m). \tag{1}$$

Now take any oriented $(r+1)$-cell τ. Its boundary is an r-chain $\sigma_1 + \cdots + \sigma_m$. We wish the sum of the corresponding r-vectors to be 0:

$$\{\sigma_1\} + \cdots + \{\sigma_m\} = 0 \quad \text{if} \quad \sigma_1 + \cdots + \sigma_m = \partial\tau \text{ for some } \tau. \tag{2}$$

Requiring all such relations to hold turns $\mathbf{S}_r$ into a linear space $\mathbf{T}_r$. We shall call any element of $\mathbf{T}_r$ an *r-vector*. (Strictly speaking, let $\mathbf{S}_r'$ denote the linear subspace of $\mathbf{S}_r$ generated by all such elements $\{\sigma_1\} + \cdots + \{\sigma_m\}$; then $\mathbf{T}_r$ is the quotient space.) We now turn $D_X(p)$ into a linear function in the space $\mathbf{T}_r$, by letting its value on an element of $\mathbf{T}_r$ be its value on any corresponding element of $\mathbf{S}_r$. To show that this is possible, we must prove the following relation:

$$D_X(p)\cdot\{\sigma_1\} + \cdots + D_X(p)\cdot\{\sigma_m\} = 0 \quad \text{if} \quad \sigma_1 + \cdots + \sigma_m = \partial\tau. \tag{3}$$

We may suppose p is in τ. Given $\lambda > 0$, let us contract E^n towards p by the factor λ; then τ becomes τ_λ and σ_i becomes $\sigma_{\lambda i}$, and we have

$$\partial\tau_\lambda = \sum \sigma_{\lambda i}, \qquad |\tau_\lambda| = \lambda^{r+1}|\tau|, \qquad |\sigma_{\lambda i}| = \lambda^r |\sigma_i|.$$

Take a sequence $\lambda_1, \lambda_2, \cdots \to 0$. We may use $\sigma_{\lambda_1 i}, \sigma_{\lambda_2 i}, \cdots$ in (7.2); with the help of (7.1), we find

$$D_X(p)\cdot\{\sigma_i\} = \lim_{k\to\infty} \frac{|\sigma_i|}{|\sigma_{\lambda_k i}|} X\cdot\sigma_{\lambda_k i} = \lim_{k\to\infty} \frac{1}{\lambda_k^r} X\cdot\sigma_{\lambda_k i}.$$

Also, by (H_2'),

$$\left|\frac{1}{\lambda_k^r}\sum_i X\cdot\sigma_{\lambda_k i}\right| = \frac{1}{\lambda_k^r}|X\cdot\partial\tau_{\lambda_k}| \leqq \frac{1}{\lambda_k^r} N_2 |\tau_{\lambda_k}| = \lambda_k N_2 |\tau|.$$

These relations give (3).

9. Grassmann algebra. We shall find a special manner of writing elements of $\mathbf{T}_r$. Take any ordered set $(v_1, \cdots, v_r)$ of independent vectors. Let σ be the parallelepiped with a point p as vertex and with these vectors along the edges from p, oriented by the v_i. We define the symbol $v_1 \vee \cdots \vee v_r$ by

(1) $$v_1 \vee \cdots \vee v_r = \{\sigma\}.$$

Now any element of $\mathbf{T}_r$ can be written as a sum of such elements.

From the properties of § 7, we see that the "product" $v_1 \vee \cdots \vee v_r$ is skew symmetric:

(2) $$v_2 \vee v_1 = -v_1 \vee v_2, \quad \text{and hence} \quad v \vee v = 0.$$

Also

(3) $$v_1 \vee \cdots \vee (av_i) \vee \cdots \vee v_r = a(v_1 \vee \cdots \vee v_i \vee \cdots \vee v_r).$$

We shall prove that it is linear in v_1:

(4) $$(v_1 + v_1') \vee v_2 \vee \cdots \vee v_r = v_1 \vee v_2 \vee \cdots \vee v_r + v_1' \vee v_2 \vee \cdots \vee v_r;$$

hence it is linear in all the v_i.

If v_1' is in the r-plane determined by $v_1, \cdots, v_r$, this is a simple geometric fact about addition of volumes. We assume this is not the case, and consider a few values of r.

For $r = 1$, a 1-vector is now represented by a vector; the 1-vector $\{pq\}$ of the oriented segment pq is represented by the vector $q - p$. Relation (4) says that addition of 1-vectors (appearing on the right) is equivalent to addition of vectors (appearing on the left). We show this as follows. Choose a point p_0, and define the points and triangle

$$p_1 = p_0 + v_1, \qquad p_2 = p_1 + v_1' = p_0 + (v_1 + v_1'), \qquad \sigma = p_0p_1p_2.$$

By (8.2),

$$\{p_0p_1\} + \{p_1p_2\} + \{p_2p_0\} = 0, \quad \text{hence} \quad \{p_0p_2\} = \{p_0p_1\} + \{p_1p_2\};$$

this is the required relation.

For $r = 2$, take the above points p_i, and also the points $q_i = p_i + v_2$ $(i = 0, 1, 2)$. Set $\sigma' = q_0q_1q_2$. The p_i and q_i are vertices of a 3-cell τ whose faces are σ and σ', and also three parallelograms $\sigma_{01}, \sigma_{12}, \sigma_{02}$, where σ_{ij} contains p_ip_j. We have clearly

$$\{\sigma\} = \{\sigma'\} = \tfrac{1}{2}v_1 \vee v_1',$$

$$\{\sigma_{01}\} = v_1 \vee v_2, \qquad \{\sigma_{12}\} = v_1' \vee v_2, \qquad \{\sigma_{02}\} = (v_1 + v_1') \vee v_2.$$

With proper regard to orientations, we see that

$$\partial\tau = \sigma' - \sigma + \sigma_{01} + \sigma_{12} - \sigma_{02};$$

applying (8.2) gives (4) for $r = 2$. The general case is similar.

The set of all vectors in E^n forms a vector space $V = V(E^n)$. Let $e_1, \cdots, e_n$ be a base in V. Then any vector v can be written uniquely as $\sum v^i e_i$; the v^i are the components of v. Since we consider 1-vectors and vectors as the same, the e_i form a base in $\mathbf{T}_1$. For $r = 2$, using (2), (3) and (4) gives

(5) $$v_1 \vee v_2 = \sum_{i,j=1}^{n} v_1^i v_2^j (e_i \vee e_j) = \sum_{i<j} \begin{vmatrix} v_1^i & v_2^i \\ v_1^j & v_2^j \end{vmatrix} e_i \vee e_j.$$

It may be shown that the $e_{ij} = e_i \vee e_j$ $(i < j)$ are independent in $\mathbf{T}_2$; by (5), they form a base in $\mathbf{T}_2$. Similarly, the $e_{\lambda_1 \cdots \lambda_r} = e_{\lambda_1} \vee \cdots \vee e_{\lambda_r}$ $(\lambda_1 < \cdots < \lambda_r)$ form a base in $\mathbf{T}_r$, and any r-vector α can be written uniquely in the form

(6) $$\alpha = \sum_{\lambda_1 < \cdots < \lambda_r} \alpha^{\lambda_1 \cdots \lambda_r} e_{\lambda_1 \cdots \lambda_r}.$$

The $\alpha^{\lambda_1 \cdots \lambda_r}$ are the "components" of α.

It follows that $\mathbf{T}_r$ is of dimension $\binom{n}{r}$. In particular, $\mathbf{T}_n$ is of dimension 1, with base element $e_{1\cdots n}$, and each $\mathbf{T}_k$ for $k > n$ contains the zero element only.

Through the definition

(7) $$(v_1 \vee \cdots \vee v_r) \vee (v_{r+1} \vee \cdots \vee v_{r+s}) = v_1 \vee \cdots \vee v_{r+s},$$

we have a bilinear multiplication between $\mathbf{T}_r$ and $\mathbf{T}_s$, with product in $\mathbf{T}_{r+s}$. If we include the space $\mathbf{T}_0$ = the real numbers, the system of the $\mathbf{T}_r$ with these operations is the *Grassmann algebra* of V. Let us denote $\mathbf{T}_r$ by $V_{[r]}$.

For $n \leqq 3$, any r-vector α (for any $r > 0$) equals $\{\sigma\}$ for some oriented r-cell σ. This is not the case for $n \geqq 4$; for instance, $e_{12} + e_{34}$ cannot be written in this form. Any α of the form $\{\sigma\}$ is a *simple* r-vector.

10. The dual algebra. The set of linear functions f in a vector space V forms a vector space, with the following definitions: the function af has the value $af(v)$ at v; the function $f+g$ has the value $f(v)+g(v)$ at v. This space is the *conjugate space* $\bar{V}$ of V.

Since $D_X(p)$ is a linear function in the vector space $\mathbf{T}_r = V_{[r]}$, we may consider it as being an element of the conjugate space of $V_{[r]}$; we denote this conjugate space by $V^{[r]}$. We call its elements *r-covectors*; the elements of $\bar{V} = V^{[1]}$ are *covectors*. We shall find a special manner of representing elements of $V^{[r]}$.

Let $e_1, \cdots, e_n$ be a base in V. Setting

$$(1) \qquad e^i\cdot v = e^i\cdot(v^1 e_1 + \cdots + v^n e_n) = v^i$$

defines a linear function e^i in V, i.e. an element of $\bar{V}$. The elements $e^1, \cdots, e^n$ are easily seen to form a base in the base *dual* to the e_i. Now any element f of $\bar{V}$ may be written uniquely as $\sum f_i e^i$; the f_i are the components of f. Hence also $\bar{V}$ and V are of the same dimension; since $V^{[r]}$ and $V_{[r]}$ are conjugate, they are also of the same dimension.

The e^i may be defined by the relation

$$(2) \qquad e^i\cdot e_j = \delta^i_j = 1 \quad \text{if} \quad i = j, \qquad = 0 \quad \text{if} \quad i \neq j.$$

Let us define base elements $e^{\lambda_1\cdots\lambda_r}$ in $V^{[r]}$ by the same formula:

$$(3) \qquad e^{\lambda_1\cdots\lambda_r}\cdot e_{\lambda_1\cdots\lambda_r} = 1 \qquad (\lambda_1 < \cdots < \lambda_r),$$

and $e^{\lambda_1\cdots\lambda_r}\cdot e_{\mu_1\cdots\mu_r} = 0$ for other μ_i $(\mu_1 < \cdots < \mu_r)$.

As a consequence of (2), we have

$$(4) \qquad f(v) = f\cdot v = \sum_{i,j=1}^{n} f_i v^j e^i\cdot e_j = \sum_{i=1}^{n} f_i v^i;$$

similarly, writing

$$(5) \qquad \xi = \sum_{\lambda_1<\cdots<\lambda_r} \xi_{\lambda_1\cdots\lambda_r} e^{\lambda_1\cdots\lambda_r},$$

the $\xi_{\lambda_1\cdots\lambda_r}$ are the components of ξ, and we have, using (9.6),

$$(6) \qquad \xi(\alpha) = \xi\cdot\alpha = \sum_{\lambda_1<\cdots<\lambda_r} \xi_{\lambda_1\cdots\lambda_r}\alpha^{\lambda_1\cdots\lambda_r}.$$

Of course a change of base results in a change in components. For $n = 3$, $r = 2$, (5), (9.6) and (6) read

$$(7) \qquad \xi = \xi_{12}e^{12} + \xi_{13}e^{13} + \xi_{23}e^{23}, \qquad \alpha = \alpha^{12}e_{12} + \alpha^{13}e_{13} + \alpha^{23}e_{23},$$

$$(8) \qquad \xi\cdot\alpha = \xi_{12}\alpha^{12} + \xi_{13}\alpha^{13} + \xi_{23}\alpha^{23}.$$

We wish to define expressions like $f^1\vee\cdots\vee f^r$, the f^i being in $\bar{V}$ and the result being in $V^{[r]}$. We wish this multiplication to be skew symmetric

and linear in each variable, and we wish to have $e^{\lambda_1}\vee\cdots\vee e^{\lambda_r} = e^{\lambda_1\cdots\lambda_r}$. This determines the multiplication. For instance, for $r = 2$,

$$(9) \qquad f\vee g = \sum_{i,j=1}^{n} f_i g_j e^i \vee e^j = \sum_{i<j} \begin{vmatrix} f_i & f_j \\ g_i & g_j \end{vmatrix} e^{ij}.$$

The value of $(f^1\vee\cdots\vee f^r)\cdot(v_1\vee\cdots\vee v_r)$ is the determinant with the elements $f^i\cdot v_j$. For instance, for $r = 2$,

$$(10) \qquad (f\vee g)\cdot(v\vee w) = \begin{vmatrix} f\cdot v & f\cdot w \\ g\cdot v & g\cdot w \end{vmatrix}.$$

To show this, we note that both sides of (10) are skew symmetric and linear in the quantities v, w; hence it is sufficient to prove this in the particular case $v = e_k$, $w = e_l$, $k < l$. But this follows at once from (9) and (3). Note that if we use (9) and (9.5) to obtain $(f\vee g)\cdot(v\vee w)$, comparing with (10) gives the Lagrange identity

$$(11) \qquad \begin{vmatrix} \sum f_i v^i & \sum f_i w^i \\ \sum g_i v^i & \sum g_i w^i \end{vmatrix} = \sum_{i<j} \begin{vmatrix} f_i & f_j \\ g_i & g_j \end{vmatrix} \begin{vmatrix} v^i & v^j \\ w^i & w^j \end{vmatrix}.$$

11. Integration of differential forms. A *differential r-form* ω in E^n is a function whose values $\omega(p)$ are r-covectors. Hence, for any p and r-vector α, $\omega(p)\cdot\alpha$ is a real number. If ω is continuous, we may define its integral over any oriented r-cell σ in E^n as follows. Take a fine subdivision of σ into oriented r-cells $\sigma_1, \cdots, \sigma_s$; choose a point p_i in each σ_i; form the sum

$$(1) \qquad \sum \omega(p_i)\cdot\{\sigma_i\};$$

take the limit of this, using a sequence of subdivisions with diameters of cells approaching 0. *It is not necessary that E^n be metric or oriented.*

For a sharp r-cochain, D_X is a continuous differential form, and

$$(2) \qquad X\cdot\sigma = \int_\sigma D_X = \lim \sum D_X(p_i)\cdot\{\sigma_i\}.$$

For if P is the r-plane of σ, oriented like σ, and we choose a metric in E^n, we may apply (7.2). See (V, 10).

B. SOME CLASSICAL TOPICS

12. Grassmann algebra in metric oriented n-space. In metric E^n, scalar products $u\cdot v$ of vectors are defined. Take any vector u. The function $\phi_u(v) = u\cdot v$ (we write also $\phi_u\cdot v$) of vectors v is linear; hence ϕ_u is a definite element of the conjugate space $\bar{V}$ of V. This is easily shown to define an isomorphism between V and $\bar{V}$. Thus any linear function ψ in V can be written in the form $\psi(v) = u\cdot v$ for a unique vector u. Since $V_{[r]}$

and $V^{[r]}$ are conjugate for each r, a definite choice of metric in $V_{[r]}$ (see (I, 12.1)) establishes a definite isomorphism between $V_{[r]}$ and $V^{[r]}$.

Now take E^n to be oriented also. Then there is a definite "unit n-vector" α_0, which equals $\{\sigma\}$ for any n-cell σ of unit volume, oriented like E^n. For any orthonormal base $(e_1, \cdots, e_n)$ (consisting of perpendicular unit vectors defining the given orientation of E^n), we clearly have

$$(1) \qquad \alpha_0 = e_{1\cdots n} = e_1 \vee \cdots \vee e_n.$$

There is also a definite "unit n-covector" ω_0 such that

$$(2) \qquad \omega_0 = e^{1\cdots n} = e^1 \vee \cdots \vee e^n, \qquad \omega_0 \cdot \alpha_0 = 1.$$

Given any r-vector α, set

$$(3) \qquad \Phi(\beta) = \omega_0 \cdot [\alpha \vee \beta], \qquad \text{all } (n-r)\text{-vectors } \beta.$$

Now Φ is a linear function in $V_{[n-r]}$ and hence is a definite element of $V^{[n-r]}$. Thus we have a definite isomorphism between $V_{[r]}$ and $V^{[n-r]}$.

13. The same, $n = 3$. Applying the results of the last section shows that in metric oriented E^3, all spaces $V_{[r]}$ and $V^{[r]}$ are isomorphic in a definite way either to $V_{[1]} = V$ or to $V_{[0]} =$ the reals. The isomorphism between $V = V_{[1]}$ and $\bar{V} = V^{[1]}$ is given by the scalar product; we consider the isomorphisms between V and $V^{[2]}$ and between V and $V_{[2]}$.

Take any vector v; set

$$(1) \qquad \Psi_v(\alpha) = \omega_0 \cdot (v \vee \alpha), \qquad \text{all 2-vectors } \alpha.$$

Since Ψ_v is a linear function of α, it is a 2-covector.

Next, take any 2-vector α; set

$$(2) \qquad \Theta_\alpha \cdot w = \omega_0 \cdot (\alpha \vee w), \qquad \text{all vectors } w.$$

Since this function of w is linear, it is given by the scalar product of w by a definite vector Θ_α.

We use this to define the *vector product* of vectors:

$$(3) \qquad u \times v = \Theta_{u \vee v};$$

thus $u \times v$ is defined by

$$(4) \qquad (u \times v) \cdot w = \omega_0 \cdot (u \vee v \vee w), \qquad \text{all vectors } w.$$

Since all operations on the right of (4) are linear, $u \times v$ is a bilinear function of u and v. Since $v \vee u = -u \vee v$ and $u \vee u = 0$, we have

$$(5) \qquad v \times u = -u \times v, \qquad u \times u = 0.$$

Let (e_1, e_2, e_3) be an orthonormal base giving the orientation chosen in E^3. Let each of u, v, w be each of e_1, e_2, e_3 in (4) in turn. As one example, we have

$$(6) \qquad (e_1 \times e_3) \cdot e_2 = \omega_0 \cdot (e_1 \vee e_3 \vee e_2) = -\omega_0 \cdot \alpha_0 = -1.$$

In this manner we see that

$$(7) \qquad e_1 \times e_2 = e_3, \qquad e_1 \times e_3 = -e_2, \qquad e_2 \times e_3 = e_1.$$

Working out $u \times v = \sum_{i,j} u^i v^j e_i \times e_j$ gives

$$(8) \qquad u \times v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} e_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_3.$$

We shall find a geometric interpretation of the vector product. Given u and v, choose an orthonormal base (e_1, e_2, e_3) orienting E^3 properly so that u is in the e_1-direction and v is in the plane of e_1 and e_2, on the same side of e_1 that e_2 is. Say

$$u = ae_1, \qquad v = b'e_1 + be_2; \qquad \text{then } a \geqq 0, \qquad b \geqq 0.$$

Then applying (7) gives $u \times v = abe_3$. Thus, if u and v are independent, in which case a and b are $\neq 0$, then $u \times v$ is a vector perpendicular to both, and oriented so that $(u, v, u \times v)$ gives the orientation of E^3. Otherwise, $u \times v = 0$. Let σ be the parallelogram with a point p as vertex and sides along u and v. Then $|\sigma| = ab$, and hence the length of $u \times v$ is

$$(9) \qquad |u \times v| = |\sigma|.$$

We shall find the components of the 2-covector Ψ_v in (1) in an orthonormal coordinate system. Using (10.5) and (10.3), we have for instance

$$(\Psi_v)_{13} = \Psi_v(e_{13}) = \omega_0 \cdot \left(\sum v^i e_i \vee e_{13} \right) = \omega_0 \cdot v^2 e_2 \vee e_1 \vee e_3 = -v^2.$$

We find

$$(10) \qquad \Psi_v = v^3 e_{12} - v^2 e_{13} + v^1 e_{23}.$$

Similarly, for the vector Θ_α in (2),

$$(11) \qquad \Theta_\alpha = \alpha^{23} e_1 - \alpha^{13} e_2 + \alpha^{12} e_3.$$

14. The differential of a mapping. Let f be a mapping of E^n (or of an open set in E^n) into E^m which is smooth; that is, with coordinate systems in the spaces, the first partial derivatives of each component of the mapping function exist and are continuous. The *differential* ∇f of f is a concept which may be used in place of these partial derivatives, as follows. Take any point p and any vector v in E^n; then for each real number $t > 0$, $p + tv$ is a point in E^n. We set

$$(1) \qquad \nabla_v f(p) = \nabla f(p, v) = \lim_{t \to 0+} \frac{1}{t} [f(p + tv) - f(p)].$$

This is a vector in E^m, tangent to the curve which is the image under f of the line through p in the direction of v. For each p, we have a function $\nabla f(p)$ mapping vectors in E^n into vectors in E^m; it is elementary to show that this function is linear.

If $m = 1$ and E^m is the space $\mathfrak{A}$ of real numbers, we have a real valued function ϕ in E^n; with coordinates $(x^1, \cdots, x^n)$ in E^n and corresponding vectors $e_1, \cdots, e_n$ along the axes at the point p, we clearly have

$$(2) \qquad \nabla_{e_i}\phi(p) = \frac{\partial\phi(p)}{\partial x^i}, \qquad i = 1, \cdots, n.$$

In this case, $\nabla_v\phi(p)$ is a real linear function of vectors v, and thus $\nabla\phi(p)$ is a covector. Hence $\nabla\phi$ is a differential 1-form in E^n, called the *gradient* of ϕ.

We return to the general case. Given p and vectors $v_1, \cdots, v_r$ in E^n, set

$$(3) \qquad \nabla f(p, v_1\vee\cdots\vee v_r) = \nabla f(p, v_1)\vee\cdots\vee\nabla f(p, v_r);$$

this defines a linear transformation of r-vectors in E^n into r-vectors in E^m, which we also call $\nabla f(p)$.

Now let ω be any r-form in E^m. Take any point $q = f(p)$ in E^m. Then $\omega(q)\cdot\alpha'$ is a linear function of r-vectors α' in E^m; hence $\omega(q)\cdot\nabla f(p, \alpha)$ is a linear function of r-vectors α in E^n, and is thus an r-covector in E^n, which we call $(f^*\omega)(p)$. Now $f^*\omega$ is an r-form in E^n. The definition is given by

$$(4) \qquad (f^*\omega)(p)\cdot\alpha = \omega(f(p))\cdot\nabla f(p, \alpha), \qquad \text{all } r\text{-vectors } \alpha \text{ in } E^n.$$

Because of (3), we find, for differential forms ω and ξ in E^m,

$$(5) \qquad f^*(\omega\vee\xi) = f^*\omega\vee f^*\xi.$$

15. Jacobians. Let f be a smooth mapping of E^n (assumed metric and oriented) into E^m. Then the image under $\nabla f(p)$ of the unit n-vector α_0 of E^n is an n-vector in E^m, which we call the *Jacobian* $J_f(p)$ of f at p. Take for instance $n = 2$, $m = 3$, and let (e_1, e_2) be an orthonormal base in E^2. The images from p of these vectors are

$$(1) \qquad w_1(p) = \nabla f(p, e_1), \qquad w_2(p) = \nabla f(p, e_2),$$

and the Jacobian at p is the 2-vector

$$(2) \qquad J_f(p) = \nabla f(p, \alpha_0) = \nabla f(p, e_1\vee e_2) = w_1(p)\vee w_2(p).$$

If $J_f(p)$ is $\neq 0$, then $w_1(p)$ and $w_2(p)$ are independent, which clearly implies that f is one-one in a neighborhood of p; in this case the image of a neighborhood of p is a smooth piece S of surface in E^3.

Suppose f maps E^n into itself. Then $J_f(p)$ is a multiple $a\alpha_0$ of α_0; we call this number a the *algebraic Jacobian* $\mathbf{J}_f(p)$. Thus, with the unit n-covector ω_0 of E^n,

$$(3) \qquad J_f(p) = \mathbf{J}_f(p)\alpha_0, \qquad \omega_0\cdot J_f(p) = \mathbf{J}_f(p).$$

The term "Jacobian" is commonly used to denote $\mathbf{J}_f(p)$; note that $\mathbf{J}_f(p)$ is independent of the metric or orientation of E^n.

Take the case $n = 2$, $m = 3$ again. Using orthonormal coordinates (s, t) in E^2 and (x, y, z) in E^3, write the components of $\nabla_{e_1} f(p) = \partial f(p)/\partial s$ etc. as a set of three numbers. We have

$$(4) \qquad \frac{\partial f(p)}{\partial s} = \left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}\right), \qquad \frac{\partial f(p)}{\partial t} = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right).$$

Define the "Jacobian determinants"

$$\frac{\partial(x, y)}{\partial(s, t)} = \begin{vmatrix} \partial x/\partial s & \partial y/\partial s \\ \partial x/\partial t & \partial y/\partial t \end{vmatrix}, \text{ etc.}$$

Then by (9.5),

$$(5) \qquad J_f(p) = \frac{\partial(x, y)}{\partial(s, t)} e_{12} + \frac{\partial(x, z)}{\partial(s, t)} e_{13} + \frac{\partial(y, z)}{\partial(s, t)} e_{23}.$$

Define the "Jacobian vector" $\bar{J}_f(p)$ at p as that vector corresponding to the 2-vector $J_f(p)$; see (13.2). By (2) and (13.3), we may write it in the form (using orthonormal coordinates)

$$(6) \qquad \bar{J}_f(p) = \frac{\partial f(p)}{\partial s} \times \frac{\partial f(p)}{\partial t} = w_1(p)\times w_2(p).$$

By (13.11) or (13.8),

$$(7) \qquad \bar{J}_f(p) = \left(\frac{\partial(y, z)}{\partial(s, t)}, -\frac{\partial(x, z)}{\partial(s, t)}, \frac{\partial(x, y)}{\partial(s, t)}\right).$$

16. Transformation of the integral. Let ω be a uniformly continuous differential 2-form in the bounded open set R of the oriented plane E^2. We consider a smooth mapping f of the bounded open set R_0 of the space E'^2 onto R, with Jacobian $J_f \neq 0$ at all points. We wish to express $\int_R \omega$ as an integral over R_0.

Cut E'^2 into small rectangles, and let $\sigma_1, \cdots, \sigma_s$ be those lying in R_0. The image $\tau_i = f(\sigma_i)$ of σ_i is a small "curvilinear parallelogram" in R. Let p_i be a corner of σ_i, and let v_{i1}, v_{i2} be vectors on adjacent sides of σ_i, so that

$$\{\sigma_i\} = v_{i1}\vee v_{i2}.$$

Now

$$w_{i1} = \nabla f(p_i, v_{i1}), \qquad w_{i2} = \nabla f(p_i, v_{i2})$$

are side vectors of a parallelogram τ'_i which is a good approximation to τ_i if σ_i is small. By § 11, clearly

$$(1) \qquad \sum \omega(q_i)\cdot\{\tau'_i\}, \qquad q_i = f(p_i),$$

is a good approximation to $\int_R \omega$. By (14.3),

$$\{\tau'_i\} = w_{i1}\vee w_{i2} = \nabla f(p_i, v_{i1}\vee v_{i2}) = \nabla f(p_i, \{\sigma_i\}),$$

and by (14.4),

$$\sum \omega(q_i)\cdot\{\tau_i'\} = \sum \omega(f(p_i))\cdot\nabla f(p_i, \{\sigma_i\}) = \sum (f^*\omega)(p_i)\cdot\{\sigma_i\},$$

which is a good approximation to $\int_{R_0} f^*\omega$. Thus we see that

$$\int_R \omega = \int_{R_0} f^*\omega. \tag{2}$$

(The detailed proof is given in (III, 8).)

From this we shall derive the usual formula, using Jacobians, and taking $E'^2 = E^2$. With the unit 2-vector α_0 in E^2,

$$\{\tau_i'\} = \nabla f(p_i, \{\sigma_i\}) = |\sigma_i|\,\nabla f(p_i, \alpha_0) = |\sigma_i|\,J_f(p_i).$$

Let $\bar\omega$ be the real function corresponding to ω; that is, with the unit 2-covector ω_0 in E^2,

$$\omega(q) = \bar\omega(q)\omega_0, \qquad \bar\omega(q) = \omega(q)\cdot\alpha_0. \tag{3}$$

Now by (15.3),

$$\begin{aligned}\bar\omega(q_i)\,|\tau_i'| &= \omega(q_i)\cdot\{\tau_i'\}\\ &= \bar\omega(q_i)\omega_0\cdot|\sigma_i|\,J_f(p_i) = \bar\omega(q_i)J_f(p_i)\,|\sigma_i|\,.\end{aligned}$$

Summing and taking limits, we have

$$\int_R \bar\omega = \int_{R_0} \bar\omega(f(p))J_f(p)\,dp, \tag{4}$$

using the Riemann integral in both cases.

The same formulas hold in any number of dimensions. Note that neither side of (2) depends on a choice of metric.

17. Smooth manifolds. As a typical example, we take a piece S of smooth surface in E^3. At each point p of S there is a tangent plane $T(p)$. Some neighborhood U of p in $T(p)$ projects in a one-one way into S. A coordinate system in $T(p)$ projects into a coordinate system in S; this coordinate system is a smooth mapping of part of the space $\mathfrak{A}^2$ of pairs of real numbers into S. Where two such coordinate systems overlap, they are related by a smooth mapping of part of $\mathfrak{A}^2$ into itself, with non-vanishing Jacobian. This suggests the general definition of a smooth manifold, using such coordinate systems, without reference to any containing space, as in (II, 10).

With S as above, let v be a vector in $T(p)$. The points $p_t = p + tv$ in $T(p)$ project into points q_t in S (for t not too large). These points q_t form a "parametrized curve" C in S, which we may take as defining a corresponding vector "of S" at p. In place of the points p_t, we could use any function p_t' in $T(p)$ with $p_0' = p$ which has the same tangent vector at $t = 0$: $(\partial p_t'/\partial t)_{t=0} = v$; this would project into a parametrized curve C' with points q_t' in S equivalent to C. We may use the definitions of av and of $v + w$ in $T(p)$ to give corresponding definitions in S at p. The vectors in S at p now form a vector space $V(p)$. Using this vector space, we may define r-vectors and r-covectors in S at p; hence we may define differential forms in S. In a general smooth manifold (not in Euclidean space), we may define vectors at a point by means of parametrized curves, and define addition of vectors with the help of coordinate systems.

To define r-dimensional integration in an n-dimensional manifold M, one must first choose simple r-dimensional domains of integration; (rectilinear) r-cells are not defined here. One may choose oriented pieces of r-dimensional submanifolds of M for such domains. We have now the problem of defining the integral of an r-form ω over an oriented piece of an r-manifold.

Take first $n = 1$; then we have an oriented curve C, abstractly defined. It is of course equivalent to an interval of the real numbers. Now $\int_C \omega$ is defined for a 1-form ω. Suppose C is in E^3 and ω is defined in a neighborhood of C. Let $p_0, p_1, \cdots, p_m$ be a division of C into short arcs. Then the vectors $p_{i+1} - p_i$ are nearly tangent to C, and

$$\sum_{i=0}^{m-1} \omega(p_i)\cdot\{p_ip_{i+1}\} = \sum_{i=0}^{m-1} \omega(p_i)\cdot(p_{i+1} - p_i) \tag{1}$$

is an approximation to the integral $\int_C \omega$.

Take an oriented piece S of surface in E^3 again. Supposing S is a small piece, we may cut it up into small curvilinear pieces $\tau_1, \cdots, \tau_s$, for instance images $f(\sigma_1), \cdots, f(\sigma_s)$ of rectangles in a coordinate system (see the τ_i in § 16). With tangent vectors w_{i1}, w_{i2} at the vertex q_i of τ_i, we may form

$$\sum \omega(q_i)\cdot(w_{i1}\vee w_{i2}) \tag{2}$$

and use this as an approximation to $\int_S \omega$; compare (16.1). Because of (16.2), we see that we may equivalently define $\int_S \omega$ as being $\int_{S_0} f^*\omega$, if $S = f(S_0)$, S_0 being part of the Euclidean plane.

The latter definition is independent of the fact that S is in E^3. With S in E^3, if ω is defined throughout a neighborhood of S in E^3, we could use polyhedral approximations to define the integral; see Chapter X.

18. Particular forms of integrals in 3-space. Here we have r-dimensional integration for $r = 0$, 1, 2 and 3. Letting E^3 be metric and oriented, an r-covector in E^3 corresponds either to a real number or to a vector (see § 13); this gives us special forms of the integrals, which we discuss.

$r = 0$. A 0-cell is a point p (without orientation properties). A 0-covector is a real number; hence a 0-form is a real function. The